Middle East Technical University

Department of Mechanical Engineering

ME 413 Introduction to Finite Element Analysis

Chapter 3

Computer Implementation of 1D FEM

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Summary of Chapter 2

- In Chapter 2 Ritz method was improved into FEM.
  - Solution is not global anymore.
  - Solution over each element is simple. Complicated 2D and 3D solutions on complex domains can be obtained, by using necessary number of elements.
  - Problems with multiple materials can be solved.
  - Approximation function selection is very well defined and independent of BCs.

- But the procedure of Chapter 2 still have difficulties.
  - Writing approximation functions one-by-one for each node is difficult. In 2D and 3D it'll be even more difficult.
  - Approximation functions change when mesh changes.
  - Symbolic math is limited and costly.

Exercise : Modify Example2_1v2.m code so that it asks the user to enter NE and works automatically. Using `tic` & `toc` commands, measure computation time for $NE = 5,\ 10,\ 50,\ 100,\ 1000$. Improve the speed of the code in any way you can.

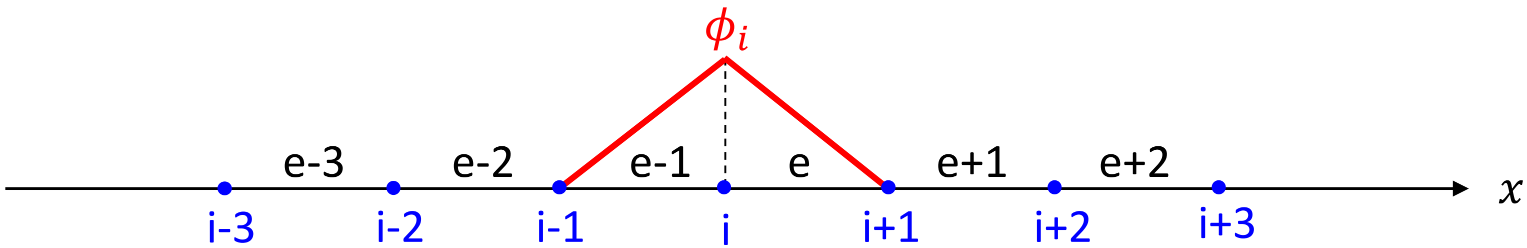
What Is This Chapter About?

- We'll improve the solution procedure of Chapter 2 so that
  - it is almost completely mesh independent
  - approximation function calculations are easy
  - symbolic calculations are avoided
  - solution is efficient and very algorithmic

- To achieve these we'll use
  - Elemental weak form
  - Master element concept
  - Gauss Quadrature numerical integration

Node Based Integral Calculation of Chapter 2

- In FEM ϕ's have local support.
- For linear elements, ϕ's are nonzero over at most two elements.

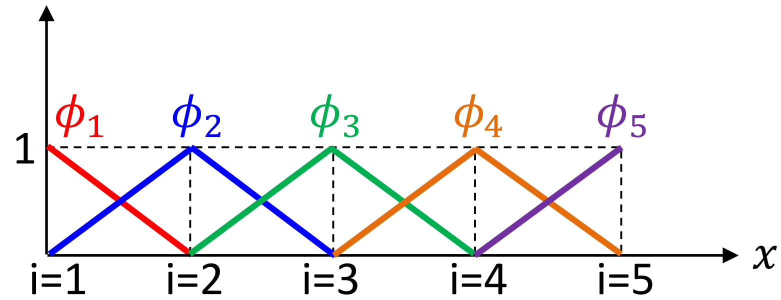


- Integral of the i^{th} eqn. contains ϕ_i in all its terms (see slide 2-12).
- But ϕ_i is nonzero only over elements e-1 and e.
- Therefore integral calculations simplify as follows

Integral of eqn. i : $$I_i = \underbrace{\int_{\Omega} f(\phi_i)\, dx}_{\text{Over the whole problem domain}} = \underbrace{\int_{\Omega^{e-1}} f(\phi_i)\, dx}_{\text{Over element e-1 only}} + \underbrace{\int_{\Omega^{e}} f(\phi_i)\, dx}_{\text{Over element e only}}$$

Node Based Integral Calculation of Chapter 2 (cont'd)

- For a 5 node mesh



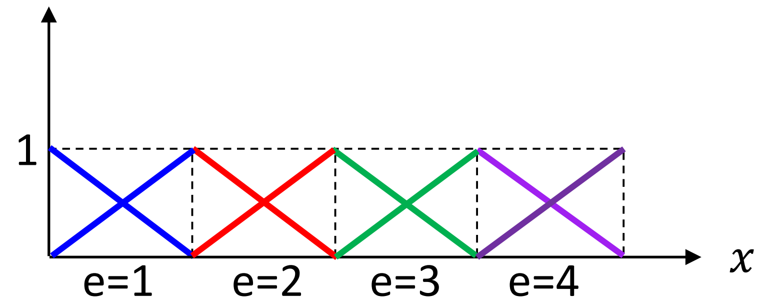
- All integrals are

$$
\left.
\begin{aligned}
I_1 &= \int_{\Omega^1} f(\phi_1)\,dx \\
I_2 &= \int_{\Omega^1} f(\phi_2)\,dx + \int_{\Omega^2} f(\phi_2)\,dx \\
I_3 &= \int_{\Omega^2} f(\phi_3)\,dx + \int_{\Omega^3} f(\phi_3)\,dx \\
I_4 &= \int_{\Omega^3} f(\phi_4)\,dx + \int_{\Omega^4} f(\phi_4)\,dx \\
I_5 &= \int_{\Omega^4} f(\phi_5)\,dx
\end{aligned}
\right\}
$$

- This is node based thinking.
- We evaluate the integral of each equation, which are associated with one node and one ϕ.
- In FEM codes we prefer element based operations.

New Element Based Integral Calculation

- Instead of thinking about each equation individually, concentrate on elements and determine the contribution of each element to each equation.



e=1 contributes to I_1 and I_2

e=2 contributes to I_2 and I_3

e=3 contributes to I_3 and I_4

e=4 contributes to I_4 and I_5

$$
\begin{aligned}
I_1 &= \int_{\Omega^1} f(\phi_1)\, dx \\
I_2 &= \int_{\Omega^1} f(\phi_2)\, dx + \int_{\Omega^2} f(\phi_2)\, dx \\
I_3 &= \int_{\Omega^2} f(\phi_3)\, dx + \int_{\Omega^3} f(\phi_3)\, dx \\
I_4 &= \int_{\Omega^3} f(\phi_4)\, dx + \int_{\Omega^4} f(\phi_4)\, dx \\
I_5 &= \int_{\Omega^4} f(\phi_5)\, dx
\end{aligned}
$$

Elemental Thinking

- For the following model DE

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+b\frac{du}{dx}+cu=f \ , \qquad 0<x<L$$

weak form is

$$\int_0^L\left(a\frac{du}{dx}\frac{dw}{dx}+bw\frac{du}{dx}+cwu\right)dx \ = \ \int_0^L wf\ dx \ + \underbrace{\left[wa\frac{du}{dx}\right]_L}_{Q_L} \ \underbrace{-\left[wa\frac{du}{dx}\right]_0}_{Q_0}$$

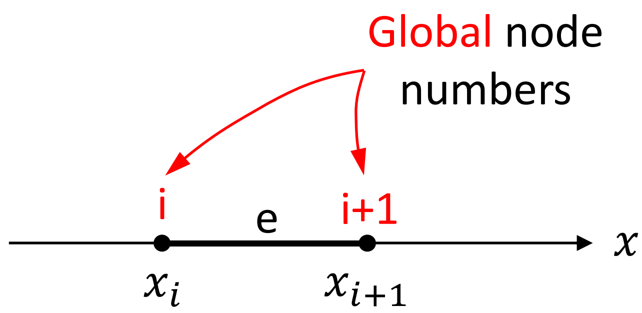
- In Chapter 2 FE solution of this resulted in a $NN \times NN$ global system

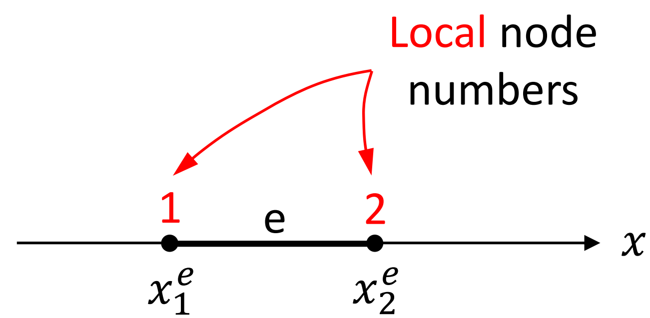
$$[K]\{u\}=\{F\}+\{Q\}$$

- The new idea is to
  - write weak form over each element individually
  - obtain elemental systems
  - add them up to get the global system

Local Node Numbering and Elemental Weak Form

- Consider the following linear element





- Elemental weak form of the model DE is

$$\int_{x_1^e}^{x_2^e} \left(a \frac{du}{dx}\frac{dw}{dx} + bw\frac{du}{dx} + cwu \right) dx = \int_{x_1^e}^{x_2^e} wf \, dx + \left[w \frac{du}{dx} \right]_{x_2^e} - \left[w \frac{du}{dx} \right]_{x_1^e}$$

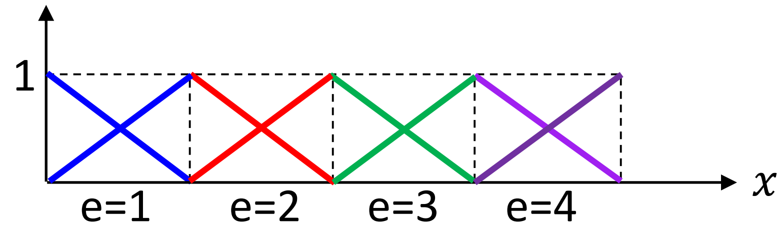
which will result in the following $NN \times NN$ elemental system

$$[K^e]\{u\} = \{F^e\} + \{Q^e\}$$

Elemental System

$$[K^e]\{u\} = \{F^e\} + \{Q^e\}$$

- Elemental systems are sparse.
- On a mesh of 4 linear elements



For e=1

$$\begin{bmatrix} \otimes & \otimes & \cdot & \cdot & \cdot \\ \otimes & \otimes & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} \otimes \\ \otimes \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \begin{Bmatrix} \otimes \\ \otimes \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} + \begin{Bmatrix} \otimes \\ \otimes \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix}$$

For e=2

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \otimes & \otimes & \cdot & \cdot \\ \cdot & \otimes & \otimes & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} \cdot \\ \otimes \\ \otimes \\ \cdot \\ \cdot \end{Bmatrix} = \begin{Bmatrix} \cdot \\ \otimes \\ \otimes \\ \cdot \\ \cdot \end{Bmatrix} + \begin{Bmatrix} \cdot \\ \otimes \\ \otimes \\ \cdot \\ \cdot \end{Bmatrix}$$

For e=3

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \otimes & \otimes & \cdot \\ \cdot & \cdot & \otimes & \otimes & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} \cdot \\ \cdot \\ \otimes \\ \otimes \\ \cdot \end{Bmatrix} = \begin{Bmatrix} \cdot \\ \cdot \\ \otimes \\ \otimes \\ \cdot \end{Bmatrix} + \begin{Bmatrix} \cdot \\ \cdot \\ \otimes \\ \otimes \\ \cdot \end{Bmatrix}$$

For e=4

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \otimes & \otimes \\ \cdot & \cdot & \cdot & \otimes & \otimes \end{bmatrix} \begin{Bmatrix} \cdot \\ \cdot \\ \cdot \\ \otimes \\ \otimes \end{Bmatrix} = \begin{Bmatrix} \cdot \\ \cdot \\ \cdot \\ \otimes \\ \otimes \end{Bmatrix} + \begin{Bmatrix} \cdot \\ \cdot \\ \cdot \\ \otimes \\ \otimes \end{Bmatrix}$$

Small Elemental Systems of Size *NEN* x *NEN*

- Each elemental system contributes to only 2 eqns of the global system.
- It is better to think of elemental systems as $NEN \times NEN$, instead of $NN \times NN$ where NEN is the number of element's nodes (=2 for linear elements)

$$[K^e]\{u^e\} = \{F^e\} + \{Q^e\}$$

Elemental stiffness matrix $NEN \times NEN$

Elemental unknown vector $NEN \times 1$

Elemental force vector $NEN \times 1$

Elemental boundary term vector $NEN \times 1$

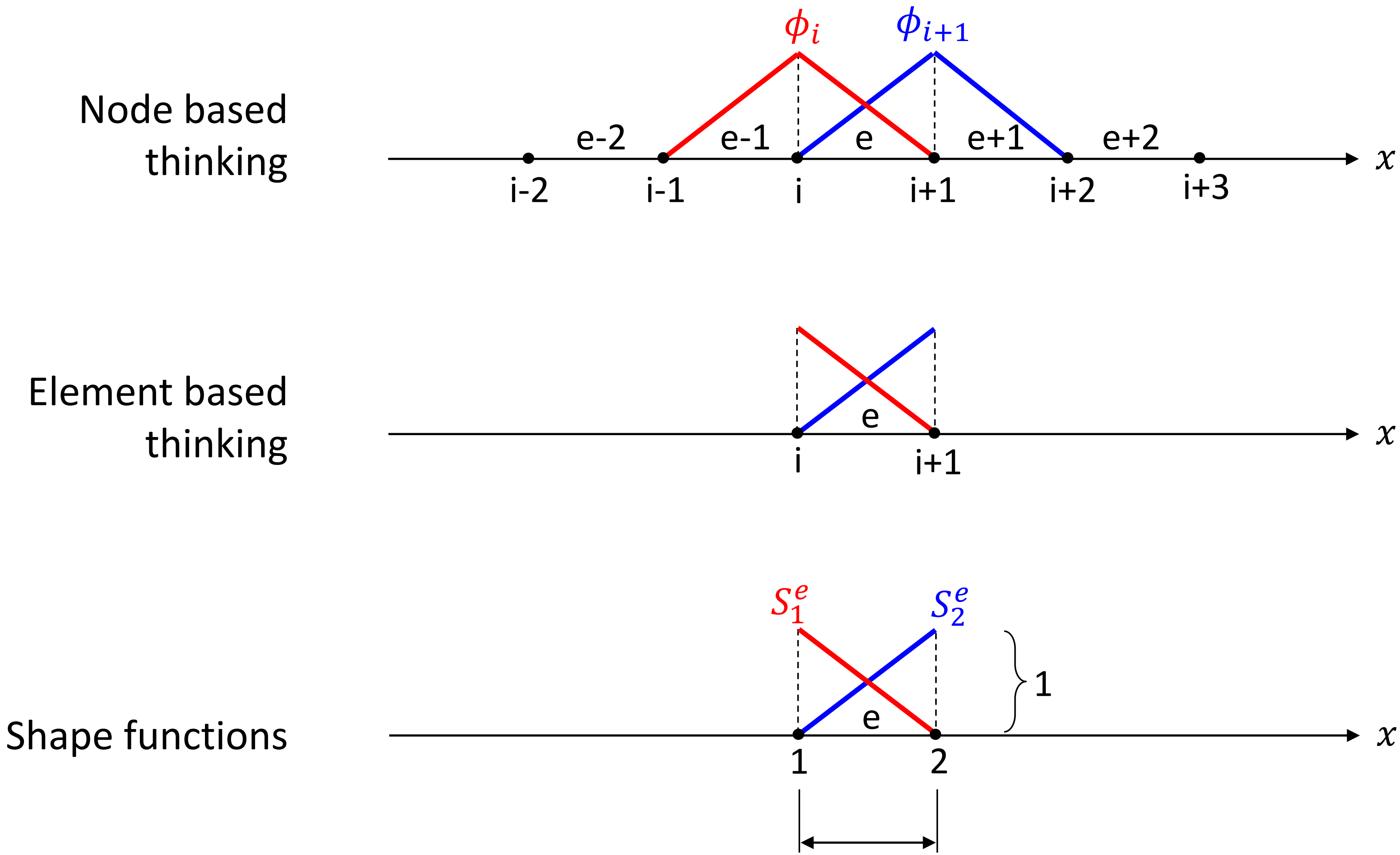
- For example for e=3, small elemental system is

$$\boxed{\begin{bmatrix} K_{11}^3 & K_{12}^3 \\ K_{21}^3 & K_{22}^3 \end{bmatrix} \begin{Bmatrix} u_1^3 \\ u_2^3 \end{Bmatrix} = \begin{Bmatrix} F_1^3 \\ F_2^3 \end{Bmatrix} + \begin{Bmatrix} Q_1^3 \\ Q_2^3 \end{Bmatrix}}$$

Assembly operation

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \otimes & \otimes & \cdot \\ \cdot & \cdot & \otimes & \otimes & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} \cdot \\ \cdot \\ \otimes \\ \otimes \\ \cdot \end{Bmatrix} = \begin{Bmatrix} \cdot \\ \cdot \\ \otimes \\ \otimes \\ \cdot \end{Bmatrix} + \begin{Bmatrix} \cdot \\ \cdot \\ \otimes \\ \otimes \\ \cdot \end{Bmatrix}$$

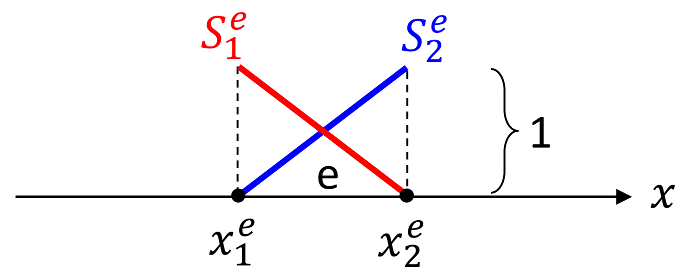
From Approximation Functions to Shape Functions



$$h^e = x_{i+1} - x_i = x_2^e - x_1^e$$

Shape Functions

- Similar to ϕ's, shape functions also have the Kronecker-delta property

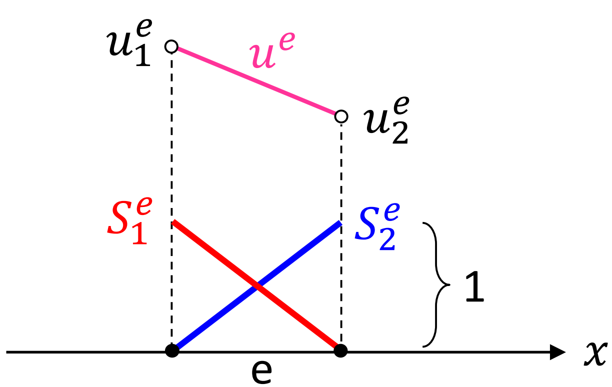


$$S_i^e(x_j^e) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

- For a linear element shape functions are

$$S_1^e = \frac{x_2^e - x}{h^e} \quad , \qquad S_2^e = \frac{x - x_1^e}{h^e}$$

- FE solution over element e is



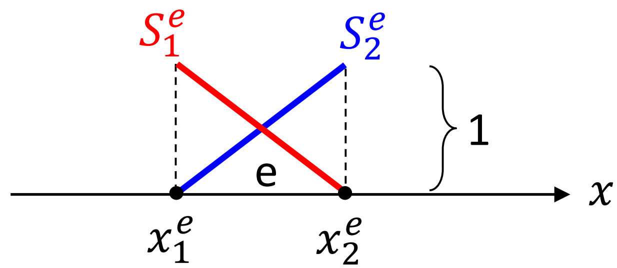
$$u^e = \sum_{j=1}^{NEN} u_j^e \, S_j^e$$

Number of element's nodes (=2 for linear elements)

Nodal unknown at element e's j^{th} node

Shape Functions (cont'd)

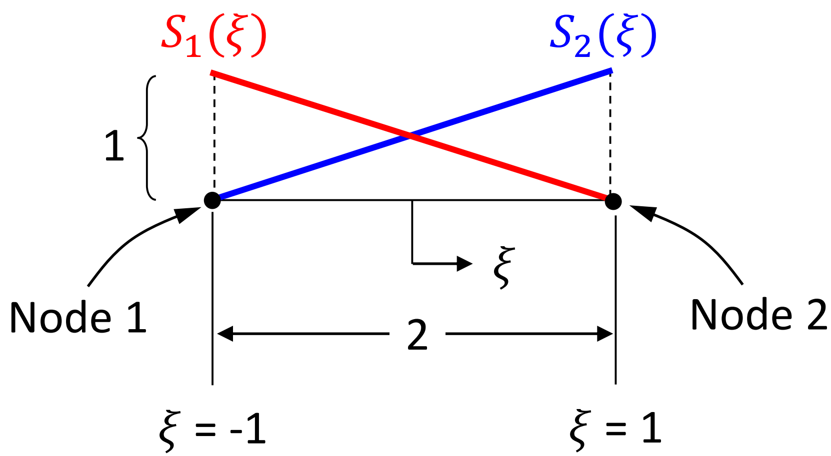
- Difficulties with these shape functions are
  - For each element they will be different functions of x.
  - Integration over an element will have limits of x_1^e and x_2^e , which are not appropriate for Gauss Quadrature integration.



- The cure is to use the concept of master element.

Master Element

- 1D, linear master element is defined using the local coordinate ξ (ksi).



$$S_1 = \frac{1-\xi}{2}$$

Superscript "e" is NOT necessary here

$$S_2 = \frac{1+\xi}{2}$$

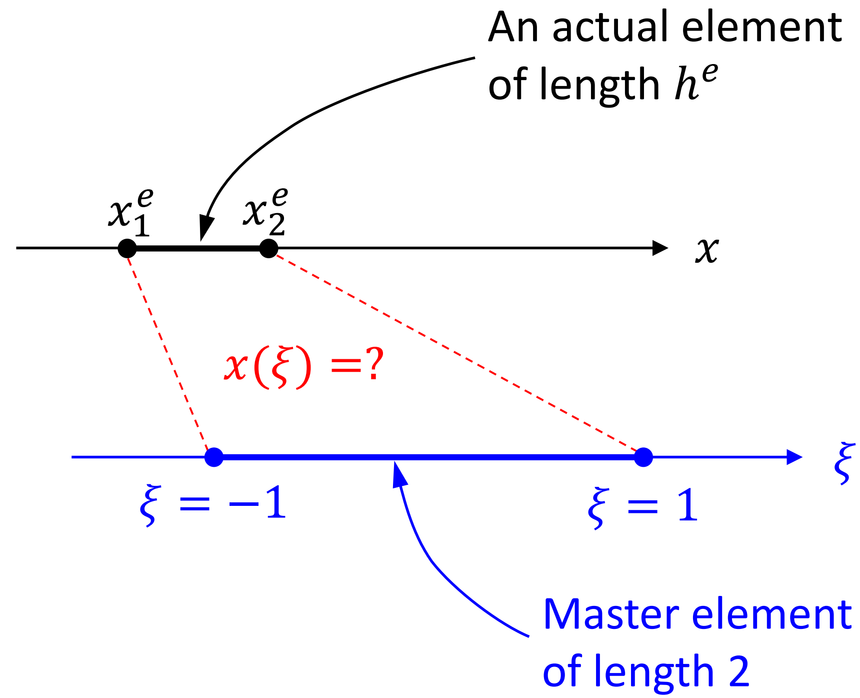
- For all linear elements in a 1D mesh, there is only a single master element.
- Master element has a length of 2.
- End points are $\xi = -1$ and $\xi = 1$, which are suitable for Gauss Quadrature.

Mapping Between an Actual Element & Master Element

- If a mesh has only linear elements, than we only need to define 2 shape functions.
- This is a great simplification, but it comes with a price.

- In order to express everything in the integrals in terms of ξ, we need to obtain the relation between the global x coordinate and the local ξ coordinate.
- This relation is linear as shown

$$x = A\xi + B$$



- Using the fact that end points of the actual element coincide with those of the master element, we get

$$\boxed{x = \frac{h^e}{2}\xi + \frac{x_1^e + x_2^e}{2}}$$

This relation is different for each and every element

Jacobian of an Element

- In the integrals of the weak form we have the first derivative of u.

$$u^e = \sum_{j=1}^{NEN} u_j^e S_j \qquad \rightarrow \qquad \frac{du^e}{dx} = \sum_{j=1}^{NEN} u_j^e \frac{dS_j}{dx}$$

- Master element shape functions are written in terms of ξ.
- Therefore x derivatives should be expressed in terms of ξ derivatives.

$$\frac{dS_j}{dx} = \frac{dS_j}{d\xi}\frac{d\xi}{dx}$$

$$\frac{dS_1}{d\xi} = -0.5 \quad , \quad \frac{dS_2}{d\xi} = 0.5$$

$= \frac{2}{h^e}$ (using the boxed equation of the previous slide.

- $J^e = \frac{dx}{d\xi} = \frac{h^e}{2}$ is the Jacobian of element e. It is the ratio of actual element's length to the length of the master element.

Example 3.1

e.g. **Example 3.1** Solve the following problem using a uniform mesh of 4 linear elements of length $h^e = 0.25$.

$$-\frac{d^2u}{dx^2} - u = -x^2 \ , \qquad 0 < x < 1$$

$$u(0) = 0 \ , \qquad u(1) = 0$$

- Elemental weak form is

$$\int_{x_1^e}^{x_2^e} \left(\frac{du}{dx}\frac{dw}{dx} - wu \right) dx = \int_{x_1^e}^{x_2^e} -wx^2 \, dx + \left[w\frac{du}{dx} \right]_{x_2^e} - \left[w\frac{du}{dx} \right]_{x_1^e}$$

- To get 2x2 elemental system of eqns, substitute the following approximate solution into the elemental weak form

$$u = \sum_{j=1}^{NEN} u_j^e S_j$$

Example 3.1 (cont'd)

$$\int_{\Omega^e}\left[\frac{d}{dx}\left(\sum u_j^e S_j\right)\frac{dw}{dx} - w\sum u_j^e S_j\right]dx = \int_{\Omega^e} -wx^2\,dx + \left[w\frac{du}{dx}\right]_{x_2^e} - \left[w\frac{du}{dx}\right]_{x_1^e}$$

- Elemental system is 2x2 and we need 2 weight functions to get it.
- In GFEM $w_1 = S_1$, $w_2 = S_2$

Eqn 1 : $$\int_{\Omega^e}\left[\frac{d}{dx}\left(\sum u_j^e S_j\right)\frac{dS_1}{dx} - S_1\sum u_j^e S_j\right]dx = \int_{\Omega^e} -S_1x^2\,dx + \underbrace{\left[\underbrace{S_1}_{0}\frac{du}{dx}\right]_{x_2^e}}_{0} \underbrace{-\left[\underbrace{S_1}_{1}\frac{du}{dx}\right]_{x_1^e}}_{Q_1^e}$$

Eqn 2 : $$\int_{\Omega^e}\left[\frac{d}{dx}\left(\sum u_j^e S_j\right)\frac{dS_2}{dx} - S_2\sum u_j^e S_j\right]dx = \int_{\Omega^e} -S_2x^2\,dx + \underbrace{\left[\underbrace{S_2}_{1}\frac{du}{dx}\right]_{x_2^e}}_{Q_2^e} \underbrace{-\left[\underbrace{S_1}_{0}\frac{du}{dx}\right]_{x_1^e}}_{0}$$

- In general the i^{th} eqn of element e is obtained by using $w = S_i$

$$\boxed{\text{Eqn i :}\quad \int_{\Omega^e}\left[\left(\sum u_j^e \frac{dS_j}{dx}\right)\frac{dS_i}{dx} - S_i\sum u_j^e S_j\right]dx = \int_{\Omega^e} -S_ix^2\,dx + Q_i^e}$$

Example 3.1 (cont'd)

- Change the integration parameter from x to ξ (refer to slide 3-16)

$dx = J^e d\xi$

Eqn i : $$\int_{-1}^{1}\left[\left(\sum u_j^e \frac{dS_j}{dx}\right)\frac{dS_i}{dx} - S_i\left(\sum u_j^e S_j\right)\right]dx = \int_{-1}^{1} -S_i x^2\, dx + Q_i^e$$

$$\frac{dS}{dx} = \frac{dS}{d\xi}\frac{1}{J^e}$$

$$x = \frac{h^e}{2}\xi + \frac{x_1^e + x_2^e}{2}$$

Eqn i : $$\int_{-1}^{1}\left[\left(\sum u_j^e \frac{dS_j}{d\xi}\frac{1}{J^e}\right)\frac{dS_i}{d\xi}\frac{1}{J^e} - S_i \sum u_j^e S_j\right]J^e d\xi = \int_{-1}^{1} S_i\, f(\xi)\, J^e d\xi + Q_i^e$$

- Take the summation sign outside the integral and take the integrand into u_j^e paranthesis.

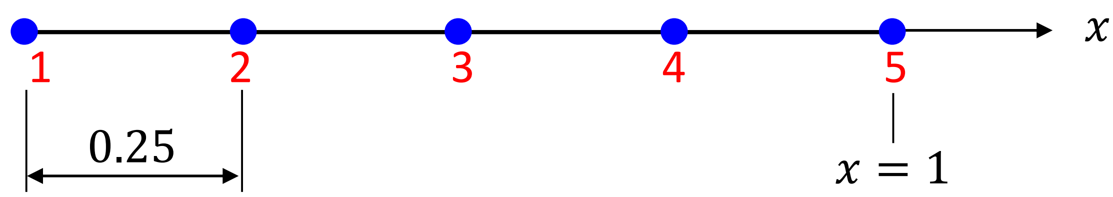
Eqn i : $$\sum \int_{-1}^{1}\left(\frac{dS_i}{d\xi}\frac{1}{J^e}\frac{dS_j}{d\xi}\frac{1}{J^e} - S_i S_j\right)J^e d\xi\, u_j^e = \int_{-1}^{1} S_i\, f(\xi)\, J^e d\xi + Q_i^e$$

Example 3.1 (cont'd)

$$\text{Eqn i :} \quad \sum \underbrace{\boxed{\int_{-1}^{1} \left(\frac{dS_i}{d\xi} \frac{1}{J^e} \frac{dS_j}{d\xi} \frac{1}{J^e} - S_i S_j \right) J^e d\xi}}_{K_{ij}^e} u_j^e = \underbrace{\boxed{\int_{-1}^{1} S_i \, f(\xi) \, J^e d\xi}}_{F_i^e} + Q_i^e$$

- Summation sign is over $j = 1, 2.$
- i index also goes from 1 to 2.
- $i = 1$ gives the first equation, $i = 2$ gives the second equation.
- 2x2 elemental system is $[K^e]\{u\} = \{F^e\} + \{Q^e\}$
- We don't need to do any calculations for Q_i^e values (Details will come).

Example 3.1 (cont'd)



- For each element $h^e = 0.25$.
- Jacobian for each element is $J^e = h^e/2 = 0.125$
- All elements are linear. Shape functions and their derivatives are

$$S_1 = \frac{1-\xi}{2} \quad , \qquad S_2 = \frac{1+\xi}{2}$$

$$\frac{dS_1}{d\xi} = -0.5 \quad , \qquad \frac{dS_2}{d\xi} = 0.5$$

- We need everything to evalute the entries of K^e and F^e one-by-one for each element.

Example 3.1 (cont'd)

$$K_{ij}^e = \int_{-1}^{1} \left(\frac{dS_i}{d\xi} \frac{1}{J^e} \frac{dS_j}{d\xi} \frac{1}{J^e} - S_i S_j \right) J^e d\xi$$

- For e=1

$$\left.\begin{aligned}
K_{11}^1 &= \int_{-1}^{1} \left(\frac{dS_1}{d\xi} \frac{1}{J^e} \frac{dS_1}{d\xi} \frac{1}{J^e} - S_1 S_1 \right) J^e d\xi = \frac{47}{12} \\
K_{12}^1 &= \int_{-1}^{1} \left(\frac{dS_1}{d\xi} \frac{1}{J^e} \frac{dS_2}{d\xi} \frac{1}{J^e} - S_1 S_2 \right) J^e d\xi = -\frac{97}{24} \\
K_{21}^1 &= K_{12}^1 \qquad \text{([}K^e\text{] is symmetric. Interchange } i \text{ \& } j \text{ and see)} \\
K_{22}^1 &= \int_{-1}^{1} \left(\frac{dS_2}{d\xi} \frac{1}{J^e} \frac{dS_2}{d\xi} \frac{1}{J^e} - S_2 S_2 \right) J^e d\xi = \frac{47}{12}
\end{aligned}\right\} \quad K^1 = \begin{bmatrix} \frac{47}{12} & -\frac{97}{24} \\ -\frac{97}{24} & \frac{47}{12} \end{bmatrix}$$

Example 3.1 (cont'd)

- No need to calculate $[K^2]$, $[K^3]$ or $[K^4]$.
- They will all be equal to $[K^1]$. This is a special case for this problem. Can you see why?

- Let's start $\{F^e\}$ calculations.

$$F_i^e = \int_{-1}^{1} S_i \, f(\xi) \, J^e d\xi$$

$$f(\xi) = -[x(\xi)]^2$$

- For e=1 :

$$f = -\left[\frac{h^e}{2}\xi + \frac{x_1^e + x_2^e}{2}\right]^2 = -\left(\frac{\xi + 1}{8}\right)^2$$

$$F_1^1 = \int_{-1}^{1} S_1 \, f(\xi) \, J^e d\xi = -\frac{1}{768} \quad , \qquad F_2^1 = \int_{-1}^{1} S_2 \, f(\xi) \, J^e d\xi = -\frac{3}{768}$$

Example 3.1 (cont'd)

- For e=2 : $f = -\left(\frac{\xi+3}{8}\right)^2$

$$F_1^2 = \int_{-1}^{1} S_1\, f(\xi)\, J^e d\xi = -\frac{11}{768} \quad , \qquad F_2^2 = \int_{-1}^{1} S_2\, f(\xi)\, J^e d\xi = -\frac{17}{768}$$

- For e=3 : f =? (find yourself)

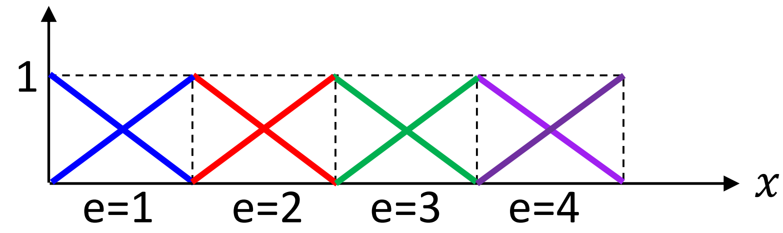
$$F_1^3 = \int_{-1}^{1} S_1\, f(\xi)\, J^e d\xi = -\frac{33}{768} \quad , \qquad F_2^3 = \int_{-1}^{1} S_2\, f(\xi)\, J^e d\xi = -\frac{43}{768}$$

- For e=4 : f =? (find yourself)

$$F_1^4 = \int_{-1}^{1} S_1\, f(\xi)\, J^e d\xi = -\frac{67}{768} \quad , \qquad F_2^4 = \int_{-1}^{1} S_2\, f(\xi)\, J^e d\xi = -\frac{81}{768}$$

Example 3.1 (cont'd)

- Four elemental systems are



For e=1 : $$\frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1^1 \\ u_2^1 \end{Bmatrix} = -\frac{1}{768}\begin{Bmatrix} 1 \\ 3 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix}$$

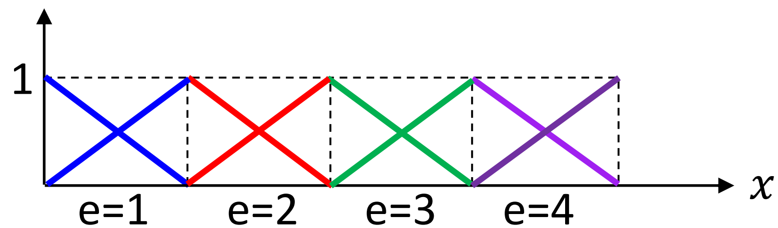
For e=2 : $$\frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1^2 \\ u_2^2 \end{Bmatrix} = -\frac{1}{768}\begin{Bmatrix} 11 \\ 17 \end{Bmatrix} + \begin{Bmatrix} Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

For e=3 : $$\frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1^3 \\ u_2^3 \end{Bmatrix} = -\frac{1}{768}\begin{Bmatrix} 33 \\ 43 \end{Bmatrix} + \begin{Bmatrix} Q_1^3 \\ Q_2^3 \end{Bmatrix}$$

For e=4 : $$\frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1^4 \\ u_2^4 \end{Bmatrix} = -\frac{1}{768}\begin{Bmatrix} 67 \\ 81 \end{Bmatrix} + \begin{Bmatrix} Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

Example 3.1 (cont'd)

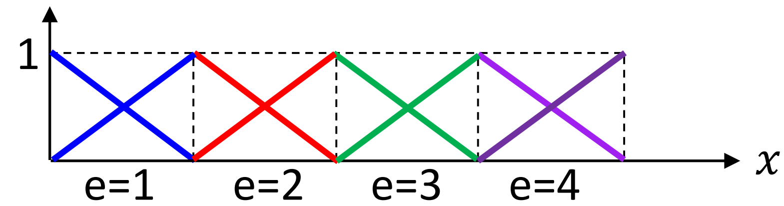
- Assemble elemental systems into 5x5 global system (see slide 3-9).



$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 & 0 \\ K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & 0 & 0 \\ 0 & K_{21}^2 & K_{22}^2 + K_{11}^3 & K_{12}^3 & 0 \\ 0 & 0 & K_{21}^3 & K_{22}^3 + K_{11}^4 & K_{12}^4 \\ 0 & 0 & 0 & K_{21}^4 & K_{22}^4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} F_1^1 \\ F_2^1 + F_1^2 \\ F_2^2 + F_1^3 \\ F_2^3 + F_1^4 \\ F_2^4 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

Example 3.1 (cont'd)

- Put the numbers in to get



$$\frac{1}{24}\begin{bmatrix} 94 & -97 & & & \\ -97 & 94+94 & -97 & & \\ & -97 & 94+94 & -97 & \\ & & -97 & 94+94 & -97 \\ & & & -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = -\frac{1}{768}\begin{Bmatrix} 1 \\ 3+11 \\ 17+33 \\ 43+67 \\ 81 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

- Balance of secondary variables :

$$Q_2^1 + Q_1^2 = \left(\frac{du}{dx}\right)_{x_2^1} + \left(-\frac{du}{dx}\right)_{x_1^2} = 0$$

$$Q_2^2 + Q_1^3 = \left(\frac{du}{dx}\right)_{x_2^2} + \left(-\frac{du}{dx}\right)_{x_1^3} = 0$$

$$Q_2^3 + Q_1^4 = \left(\frac{du}{dx}\right)_{x_2^3} + \left(-\frac{du}{dx}\right)_{x_1^4} = 0$$

Example 3.1 (cont'd)

- Global system is

$$\frac{1}{24}\begin{bmatrix} 94 & -97 & & & \\ -97 & 188 & -97 & & \\ & -97 & 188 & -97 & \\ & & -97 & 188 & -97 \\ & & & -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = -\frac{1}{768}\begin{Bmatrix} 1 \\ 14 \\ 50 \\ 110 \\ 81 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ 0 \\ 0 \\ 0 \\ Q_5 \end{Bmatrix}$$

- u_1 and u_5 are known.
- Reduce the system by dropping the 1st and 5th equations.

$$\frac{1}{24}\begin{bmatrix} 188 & -97 & \\ -97 & 188 & -97 \\ & -97 & 188 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = -\frac{1}{768}\begin{Bmatrix} 14 \\ 50 \\ 110 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

- FE solution is

$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} -0.0232 \\ -0.0405 \\ -0.0392 \end{Bmatrix}$$

Apply EBCs without Reduction

- Reduction is not easy to implement in a computer code.
- A simpler technique is to keep the 1st and 5th eqns, but modify them as follows

$$\begin{bmatrix} \boxed{1 \quad 0 \quad 0 \quad 0 \quad 0} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\ \boxed{0 \quad 0 \quad 0 \quad 0 \quad 1} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} U_1 \\ 0 \\ 0 \\ 0 \\ U_5 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Equate diagonal entries to 1, and non-diagonal entries to zero

These are the specified values of u_1 and u_5

Equate unknown Q's to zero.

- Disadvantages are
  - symmetry of [K] is lost.
  - an unnecessarily large system is solved.

Apply EBCs without Reduction (cont'd)

- A third alternative for EBCs modifies 1st and 5th eqns as follows

$$\begin{bmatrix} \boxed{L \times K_{11}} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & \boxed{L \times K_{55}} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} \boxed{L \times K_{11} \times U_1} \\ F_2 \\ F_3 \\ F_4 \\ \boxed{L \times K_{55} \times U_5} \end{Bmatrix} + \begin{Bmatrix} \boxed{0} \\ 0 \\ 0 \\ 0 \\ \boxed{0} \end{Bmatrix}$$

where L is large enough number.

- If L is large enough the 1st and 5th eqns simplify to

$$LK_{11}u_1 + \text{Negligibly small terms} = LK_{11}U_1 \quad \rightarrow \quad u_1 = U_1$$

$$LK_{55}u_5 + \text{Negligibly small terms} = LK_{55}U_5 \quad \rightarrow \quad u_5 = U_5$$

- This technique preserves possible symmetry of $[K]$.

NBCs

- If a NBC is provided, the specified Q value is used in the global system.
- Similar to the Ritz method, NBCs are satisfied not exactly, but approximately.

- Be careful in determining the SV correctly.
- If a heat conduction problem is formulated starting from

$$\left.\begin{aligned} & -\frac{d}{dx}\left(kA\frac{dT}{dx}\right)+\ldots\ldots=0 \\ & \text{then} \quad Q_1=-\left(kA\frac{dT}{dx}\right)_0 \quad \text{and} \quad Q_{NN}=\left(kA\frac{dT}{dx}\right)_L \end{aligned}\right\} \text{SV is heat in Watts}$$

- If in the same problem kA is constant and dropped from the DE

$$\left.\begin{aligned} & -\frac{d}{dx}\left(\frac{dT}{dx}\right)+\ldots\ldots=0 \\ & \text{then} \quad Q_1=-\left(\frac{dT}{dx}\right)_0 \quad \text{and} \quad Q_{NN}=\left(\frac{dT}{dx}\right)_L \end{aligned}\right\} \text{SV is temperature gradient in K/m}$$

MBCs

- Put the given mixed BC into the form

$$SV = \alpha PV + \beta$$

where α and β are known values.

- Use $\alpha PV + \beta$ in the proper place of the $\{Q\}$ vector.
- Transfer αPV to the [K] matrix and leave β on the RHS of the global system.

- If a mixed BC is given at the 5th (last) node of a 4 element mesh

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ 0 \\ 0 \\ 0 \\ \boxed{\alpha u_5} + \beta \end{Bmatrix}$$

Modify K_{55} as

$K_{55} - \alpha$

Gauss Quadrature (GQ) Integration

- In FEM integrals similar to the following ones need to be evaluated

$$K_{ij}^e = \int_{-1}^{1} \left(\frac{dS_i}{d\xi} \frac{1}{J^e} \frac{dS_j}{d\xi} \frac{1}{J^e} - S_i S_j \right) J^e d\xi \quad , \qquad F_i^e = \int_{-1}^{1} S_i \, f(\xi) \, J^e d\xi$$

- The limits [-1,1] are suitable for GQ integration, which converts an integral into a summation

Number of GQ points

$$I = \int_{-1}^{1} g(\xi) \, d\xi = \sum_{k=1}^{NGP} g(\xi_k) W_k$$

GQ weights

Coordinates of GQ points

GQ Integration (cont'd)

- GQ points and weights for different NGP values are

NGP	ξ_k	W_k
1	0.0	2.0
2	$-1/\sqrt{3}$ = $-$0.577350269189626 $1/\sqrt{3}$ = 0.577350269189626	1.0 1.0
3	$-\sqrt{0.6}$ = $-$0.774596669241483 0.0 $\sqrt{0.6}$ = 0.774596669241483	5/9 = 0.555555555555555 8/9 = 0.888888888888889 5/9 = 0.555555555555555
4	$-$ 0.861136311594953 $-$ 0.339981043584856 0.339981043584856 0.861136311594953	0.347854845137454 0.652145154862546 0.652145154862546 0.347854845137454

- NGP point GQ integration can evaluate $(2\ NGP - 1)$ order polynomial functions exactly.

Example 3.2

e.g. **Example 3.2** Evaluate K_{11}^1 and F_1^1 of Example 3.1 using GQ integration.

$$K_{11}^1 = \int_{-1}^{1} \left(\frac{dS_1}{d\xi} \frac{1}{J^e} \frac{dS_1}{d\xi} \frac{1}{J^e} - S_1 S_1 \right) J^e d\xi$$

$$= \int_{-1}^{1} \left[(-0.5) \frac{1}{0.125} (-0.5) \frac{1}{0.125} - \left(\frac{1-\xi}{2} \right) \left(\frac{1-\xi}{2} \right) \right] (0.125)\, d\xi$$

$$= \int_{-1}^{1} \underbrace{\left(\frac{-\xi^2 + 2\xi + 63}{32} \right)}_{g(\xi)} d\xi$$

- Using 1 point GQ : $K_{11}^1 = 2g(0) = 3.9375$
- Using 2 point GQ : $K_{11}^1 = g\left(-\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right) = 3.9167$
- Using 3 point GQ : $K_{11}^1 = \frac{5}{9} g\left(-\sqrt{0.6}\right) + \frac{8}{9} g(0) + \frac{5}{9} g\left(\sqrt{0.6}\right) = 3.9167$

Both are exact

Example 3.2 (cont'd)

$$F_1^1 = \int_{-1}^{1} S_1 \, f(\xi) \, J^e d\xi$$

$$= \int_{-1}^{1} -\left(\frac{1-\xi}{2}\right)\left(\frac{\xi+1}{8}\right)^2 (0.125) \, d\xi$$

$$= \int_{-1}^{1} \underbrace{\left(\frac{\xi^3+\xi^2-\xi-1}{1024}\right)}_{g(\xi)} d\xi$$

- Using 1 point GQ : $F_1^1 = 2g(0) = -0.0019531$
- Using 2 point GQ : $F_1^1 = g\left(-\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right) = -0.0013021$
- Using 3 point GQ : $F_1^1 = \frac{5}{9} g\left(-\sqrt{0.6}\right) + \frac{8}{9} g(0) + \frac{5}{9} g\left(\sqrt{0.6}\right) = -0.0013021$

Both are exact